

REMARKS ON UHLENBECK'S DECOMPOSITION THEOREM

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ABSTRACT. We present a self-contained proof of Uhlenbeck's decomposition theorem for $\Omega \in L^p(\mathbb{B}^n, so(m) \otimes \Lambda^1 \mathbb{R}^n)$ for $p \in (1, n)$ with Sobolev type estimates in the case $p \in [n/2, n)$ and Sobolev-Morrey type estimates in the case $p \in (1, n/2)$. We also prove an analogous theorem in the case when $\Omega \in L^p(\mathbb{B}^n, TCO_+(m) \otimes \Lambda^1 \mathbb{R}^n)$, which corresponds to Uhlenbeck's theorem with conformal gauge group.

1. INTRODUCTION

In 2006 T. Rivière published a solution to the so-called Heinz-Hildebrandt conjecture on regularity of solutions to conformally invariant nonlinear systems of partial differential equations in dimension 2 [10]. The key tool he used was a theorem originally due to Karen Uhlenbeck, on the existence of the so-called Coulomb gauges, in which the connection of a line bundle takes particularly simple form [15]. One should also mention that Coulomb gauges appeared earlier in the theory of geometrically motivated systems of PDE, namely – in important papers of F. Hélein on regularity of harmonic mappings between manifolds ([4, 5]).

The power of Rivière's idea was in the fact that he used Uhlenbeck's theorem to antisymmetric differential forms, which *a priori* were not interpreted as connection forms, even if the problem had clear geometric motivation. Moreover, he reformulated the theorem in a language more suited for PDE applications. Simplifying, any antisymmetric matrix Ω of 1-forms on a ball

$$\Omega : \mathbb{B}^n \rightarrow so(m) \otimes \Lambda^1 \mathbb{R}^n$$

with *sufficiently small* norm can be transformed by an orthogonal change of coordinates (gauge transformation)

$$P : \mathbb{B}^n \rightarrow SO(m)$$

to an antisymmetric matrix of co-closed forms (up to a rather regular term)

$$P^{-1}dP + P^{-1}\Omega P = *d\xi,$$

$$\text{i.e. } \Omega = P(*d\xi)P^{-1} - dP P^{-1}.$$

Here ξ is an orthogonal matrix of $(n-2)$ -forms:

$$\xi : \mathbb{B}^n \rightarrow so(m) \otimes \Lambda^{n-2} \mathbb{R}^n.$$

Such a decomposition of Ω is often referred to as Uhlenbeck's decomposition.

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The original result of Uhlenbeck was stated for $\Omega \in W^{1,p}(\mathbb{B}^n)$, with $n/2 \leq p < n$. Since then, Uhlenbeck's decomposition appeared in numerous papers on nonlinear PDE's and variational problems, each time adapted to a specific system, functional spaces and dimensions:

- Rivière [10]: $\Omega \in L^2$, $n = 2$;
- Rivière & Struwe [11]: $\Omega \in L^{2,n-2}$, $n > 2$;
- Lamm & Rivière [7]: $\Omega \in W^{1,2}$, $n = 4$;
- Meyer-Rivière [8] and Tao-Tian [14]:

$$\Omega \in W^{1,2} \cap L^{4,n-4}, \quad d\Omega \in L^{2,n-4} \quad \text{for } n \geq 4;$$

- Müller & Schikorra [9]: $\Omega \in W^{1,2}$, $n = 2$;

All the proofs of the above results (with the exception of [10]) are, up to details, adaptations of the original approach of Uhlenbeck and most of them refer the reader for certain parts of the reasoning to the original paper [15]. The latter, however, is written in the language of differential geometry (the result was used there in the context of the existence theory for Yang-Mills fields). Translating the results to Rivière's setting and filling all the sketched details was not trivial, which is probably why this extremely useful result went overlooked by the PDE community for over two decades.

All the proofs naturally split into two parts:

- proving the existence of the decomposition for any sufficiently small perturbation of a co-closed form $*d\zeta$ provided certain norm of $d\zeta$ is small;
- proving that once we have Ω , P and ξ which satisfy the equation

$$P^{-1}dP + P^{-1}\Omega P = *d\xi$$

and additionally certain norms of Ω , dP and $d\xi$ are sufficiently small, then the presumed estimates hold (the norms of P and ξ are bounded in terms of the norm of Ω).

In the results mentioned above, two strategies of proving the existence of decomposition of Ω were used. The original strategy used by Karen Uhlenbeck was to solve the equation for P

$$*d * (P^{-1}dP + P^{-1}(*d\zeta + \lambda)P) = 0$$

for a given perturbation λ of some fixed co-closed form. To do this, we look for P of the form $P = e^u$, add a boundary condition on u and define the nonlinear operator

$$T(u, \lambda) = *d * (e^{-u}de^u + e^{-u}(*d\zeta + \lambda)e^u)$$

acting on appropriate Banach spaces:

$$T(u, \lambda) : \mathcal{B} \times W^{1,p} \rightarrow L^p$$

where $\mathcal{B} = W^{2,p} \cap W_o^{1,p}$. One can apply then the Implicit Function Theorem to show that $T(u, \lambda) = 0$ has a solution u_λ continuously depending on λ . To do this, one has to show that the linearization of T at $(u, \lambda) = (0, 0)$ with respect to the first argument:

$$H(u) = \Delta u + *[d\zeta, du]$$

is an isomorphism $\mathcal{B} \rightarrow L^p$. This strategy works in Sobolev spaces for $p \in [n/2; n)$, but it fails when $1 < p < n/2$.

Another strategy was used by T. Tao and G. Tian in [14]. Again, one looks for $P = e^u$ and assumes u has zero boundary data. The equation

$$*d * (P^{-1}dP + P^{-1}(*d\zeta + \lambda)P) = 0$$

is transformed into the form

$$\Delta u = *d * (du - e^{-u}de^u - e^{-u}(*d\zeta + \lambda)e^u) = *d * F(u, \zeta, \lambda).$$

Then an iteration scheme is set to provide a solution:

$$\begin{aligned} u^0 &= 0 \\ \Delta u^{k+1} &= *d * F(u^k, \zeta, \lambda). \end{aligned}$$

We apply this strategy when $1 < p < n/2$.

In 2009 A. Schikorra gave an alternative, variational proof of the existence of Uhlenbeck's decomposition ([12]). His approach was inspired by a similar variational construction of a moving frame by F. Hélein [5]. Schikorra's methods, however, provided the gauge transformation P only in $W^{1,2}$, even for $\Omega \in L^p$ with $p > 2$ (while Uhlenbeck's and Rivière's approach gave $P \in W^{1,p}$). On the other hand, his method was much simpler and allowed him to give alternative regularity proofs for systems studied by Rivière [10] and Rivière & Struwe [11].

We begin the paper by recalling the definitions and basic properties of Morrey and Morrey-Sobolev spaces we use (Section 2).

Next, we present a self-contained, complete proof of Uhlenbeck's decomposition in Rivière's form with both Sobolev- and Morrey-Sobolev type estimates, adapted to abstract setting of PDE's. Namely, if we assume $\Omega \in W^{1,p}$, then

- for $n/2 \leq p < n$: the smallness condition for Ω is in the L^n norm and estimates are given in the Sobolev norms, P and $\xi \in W^{2,p}$ (Section 3);
- for $1 < p < n/2$: the smallness condition is expressed in the suitable Morrey space and estimates for norms of P and ξ are given in Morrey-Sobolev-type spaces (Section 4).

We express the results and the proofs in the language of differential forms, which allows applications to higher-dimensional problems (c.f. [11]). Like in all the other existing proofs, we follow the original scheme of Uhlenbeck's proof; we adapt arguments used in various adaptations of Uhlenbeck's proof mentioned above, in particular in papers [11] and [14].

Finally, we study a version of Uhlenbeck's decomposition for a larger gauge group $CO_+(n)$ (i.e. conformal transformations), which gives the decomposition theorem for a larger class of matrix-valued differential forms Ω (Section 5).

2. MORREY SPACES

Let $\Omega \subset \mathbb{R}^n$ be an open and bounded set.

Recall that the Morrey space $L^{p,s}(\Omega)$ is a collection of all functions $f \in L^p(\Omega)$ such that

$$\|f\|_{L^{p,s}}^p = \sup_{x_0 \in \Omega, r > 0} \left(\frac{1}{r^s} \int_{B_r(x_0) \cap \Omega} |f(y)|^p dy \right) < \infty.$$

When $s = 0$, the Morrey space $L^{p,0}(\Omega)$ is the same as the usual Lebesgue $L^p(\Omega)$ space. When $s = n$, the dimension of the ambient space, the Morrey space $L^{p,n}(\Omega)$ is equivalent to $L^\infty(\Omega)$, due to the Lebesgue differentiation theorem. Morrey spaces are Banach spaces.

For $1 \leq p \leq q < \infty$ and $s, \sigma \geq 0$ such that $\frac{s-n}{p} \leq \frac{\sigma-n}{q}$ we have

$$L^{q,\sigma}(\Omega) \hookrightarrow L^{p,s}(\Omega),$$

in particular

$$L^{q,n-q}(\Omega) \hookrightarrow L^{p,n-p}(\Omega).$$

Definition 2.1. We define the Morrey-Sobolev space $L_k^{p,n-kp}(\Omega)$ as

$$L_k^{p,n-kp}(\Omega) = \left\{ f \in W^{k,p}(\Omega) : \nabla^j f \in L^{p,n-jp}(\Omega) \text{ for } j = 1, \dots, k \right\}$$

with the norm

$$\|f\|_{L_k^{p,n-kp}(\Omega)} = \|f\|_{L^p(\Omega)} + \sum_{j=1}^k \|\nabla^j f\|_{L^{p,n-jp}(\Omega)}$$

Note that with this definition, $f \in L_k^{p,n-kp}(\Omega)$ does not imply Morrey estimates for f itself. However, the inclusion $L_k^{p,n-kp}(\Omega) \subset BMO(\Omega)$ holds.

Definition 2.2. We say that $f \in BMO(\Omega)$ if

$$[f]_{BMO(\Omega)} = \sup_{x_0 \in \Omega, r > 0} \left(\frac{1}{r^n} \int_{B_r(x_0) \cap \Omega} |f(y) - f_{x_0,r}| dy \right) < \infty,$$

where $f_{x_0,r}$ denotes the average of f on the set $B_r(x_0) \cap \Omega$.

By the Poincaré inequality we have

$$\int_{B_r(x_0) \cap \Omega} |f - f_{x_0,r}|^p dx \leq r^p \int_{B_r(x_0) \cap \Omega} |\nabla f|^p dx,$$

thus

$$[f]_{BMO(\Omega)} \leq C \|\nabla f\|_{L^{p,n-p}(\Omega)}.$$

We say that a domain Ω is of type (A) if there exists a constant $C > 0$ such that for any $x_0 \in \Omega$ and $0 < r < \text{diam}(\Omega)$

$$|B_r(x_0) \cap \Omega| \geq Cr^n.$$

This excludes domains with infinitely sharp cusps. For such domains we have the following generalization of Sobolev's embedding theorem

Proposition 2.3 (Morrey-Sobolev embedding). *Let $\Omega \subset \mathbb{R}^n$ be of type (A). Let $1 \leq p < \infty$ and $\alpha \in (0, p)$. If $f \in W^{1,p}(\Omega)$ is such that $\nabla f \in L^{p,n-p+\alpha}$, then $f \in C^{0,\alpha/p}(\Omega)$.*

The following Gagliardo-Nirenberg type estimate is a slight adaptation of [13, Proposition 3.2], with virtually no change in the proof.

Proposition 2.4. *For any $1 < p \leq n/2$ and $\alpha \geq 0$ there exists a constant C such that for any ball $B \subset \mathbb{B}^n$ and any $f \in L_2^{p,n-2p+2\alpha}(B)$ there holds*

$$\|\nabla f\|_{L^{2p,n-2p+2\alpha}(B)}^2 \leq C \|\nabla f\|_{L^{1,n-1}(B)} (\|\nabla^2 f\|_{L^{p,n-2p+2\alpha}(B)} + \|\nabla f\|_{L^{p,n-p+\alpha}(B)}).$$

In particular, for $f \in L_2^{p,n-2p+2\alpha}(B)$ this implies

$$\begin{aligned} (2.1) \quad \|\nabla f\|_{L^{2p,n-2p+2\alpha}(B)}^2 &= C \|\nabla f\|_{L^{2p,n-2p+2\alpha}(B)}^2 \\ &\leq C \|\nabla f\|_{L^{1,n-1}(B)} \left(\|\nabla^2 f\|_{L^{p,n-2p+2\alpha}(B)} + \|\nabla f\|_{L^{p,n-p+\alpha}(B)} \right) \\ &\leq C \|f\|_{L_2^{p,n-2p+2\alpha}(B)}^2 \end{aligned}$$

3. UHLENBECK'S DECOMPOSITION, THE CASE $n/2 \leq p < n$

We first state and prove the theorem in the case $p \in (n/2, n)$, recovering the original result of Uhlenbeck. The case $p = n/2$ follows by approximation (see Corollary 3.7 at the end of this section).

Theorem 3.1. *Let $\frac{n}{2} < p < n$. There exists $\epsilon > 0$ such that for any antisymmetric matrix $\Omega \in W^{1,p}(\mathbb{B}^n, so(m) \otimes \Lambda^1 \mathbb{R}^n)$ of 1-differential forms on \mathbb{B}^n such that*

$$\|\Omega\|_{L^n} < \epsilon$$

there exist $P \in W^{2,p}(\mathbb{B}^n, SO(m))$ and $\xi \in W^{2,p} \cap W_0^{1,p}(\mathbb{B}^n, so(m) \otimes \Lambda^{n-2} \mathbb{R}^n)$ satisfying the system

$$(3.1) \quad \begin{cases} P^{-1}dP + P^{-1}\Omega P = *d\xi & \text{on } \mathbb{B}^n, \\ d*\xi = 0 & \text{on } \mathbb{B}^n, \\ \xi = 0 & \text{on } \partial\mathbb{B}^n; \end{cases}$$

and such that

$$(3.2a) \quad \|d\xi\|_{L^p} + \|dP\|_{L^p} \leq C(n, m, p) \|\Omega\|_{L^p},$$

$$(3.2b) \quad \|d\xi\|_{L^n} + \|dP\|_{L^n} \leq C(n, m) \|\Omega\|_{L^n},$$

$$(3.2c) \quad \|d\xi\|_{W^{1,p}} + \|dP\|_{W^{1,p}} \leq C(n, m, p) \|\Omega\|_{W^{1,p}}.$$

Remark 3.2. In what follows, we write, to keep the notation simple, $\Omega \in W^{1,p}(P, \xi \in W^{2,p}$ etc.), instead of $\Omega \in W^{1,p}(\mathbb{B}^n, so(m) \otimes \Lambda^1 \mathbb{R}^n)$.

We shall break the proof of Theorem 3.1 into several lemmata. Following Rivière, we introduce sets

$$V_\epsilon = \{\Omega \in W^{1,p} : \|\Omega\|_{L^n} < \epsilon\}$$

$$U_\epsilon = \{\Omega \in W^{1,p} : \|\Omega\|_{L^n} < \epsilon \text{ and there exist } P \text{ and } \xi \text{ satisfying the system (3.1) and the estimates (3.2)}\}.$$

We show that for ϵ sufficiently small the set U_ϵ is closed and open in V_ϵ , and since the latter is path connected (it is star-shaped in $W^{1,p}$), it follows that $U_\epsilon = V_\epsilon$ which proves Theorem 3.1.

Lemma 3.3. *The set U_ϵ is closed in V_ϵ .*

Proof. Suppose (Ω_k) is a sequence in U_ϵ convergent in $W^{1,p}$ to some Ω . With every Ω_k we associate P_k and ξ_k that satisfy (3.1) and estimates (3.2):

$$(3.3) \quad \begin{cases} P_k^{-1}dP_k + P_k^{-1}\Omega_k P_k = *d\xi_k & \text{on } \mathbb{B}^n, \\ d * \xi_k = 0 & \text{on } \mathbb{B}^n, \\ \xi_k = 0 & \text{on } \partial\mathbb{B}^n, \end{cases}$$

and

$$(3.4) \quad \|d\xi_k\|_{W^{1,p}} + \|dP_k\|_{W^{1,p}} \leq C(n, m)\|\Omega_k\|_{W^{1,p}}.$$

The boundary condition on ξ_k and boundedness of P_k ($|P_k| = 1$) allow us to interpret (3.4) as boundedness of P_k and ξ_k in $W^{2,p}$, since the sequence (Ω_k) being convergent, is necessarily bounded in $W^{1,p}$. We can thus assume (after passing to subsequences) that P_k and ξ_k are weakly convergent in $W^{2,p}$ to some P and ξ .

Both the boundary condition $\xi|_{\partial\mathbb{B}} = 0$ and the condition $d * \xi = 0$ are conserved when passing to the weak limit. Moreover, possibly after passing to a subsequence, we have

$$\begin{aligned} \Omega_k \rightarrow \Omega & \quad \text{in } W^{1,p} & \Rightarrow & \quad \Omega_k \rightarrow \Omega & \quad \text{in } L^{\frac{np}{n-p}}, \\ P_k \rightharpoonup P & \quad \text{in } W^{2,p} & \Rightarrow & \quad dP_k \rightharpoonup dP & \quad \text{in } L^{\frac{np}{n-p}}, \\ \xi_k \rightharpoonup \xi & \quad \text{in } W^{2,p} & \Rightarrow & \quad d\xi_k \rightharpoonup d\xi & \quad \text{in } L^{\frac{np}{n-p}}, \end{aligned}$$

and for any small $\delta > 0$, Ω_k , dP_k and $d\xi_k$ converge strongly (to Ω , P and ξ , respectively) in L^s , with $s = \frac{np}{n-p} - \delta$, in particular in L^n , since $np/(n-p) > n$. Also, P_k are uniformly bounded in L^∞ and strongly convergent, by Sobolev embedding theorem, in L^q for any q . This is enough to show the strong convergence of $\Omega_k P_k$ to ΩP in L^s ; altogether, we may pass to the strong limit in L^s in the system (3.3), showing that the equation

$$dP + \Omega P = P * d\xi \quad \text{in } \mathbb{B}^n$$

is satisfied in the sense of distributions.

The estimates (3.2) for P and ξ are obvious. □

Remark. For any P and ξ in $W^{1,p}$ that satisfy (3.3) we have that $d\xi \in W^{1,p}$ implies $P \in W^{2,p}$. Indeed, we have $dP = P * d\xi - \Omega P$, and for $p > n/2$ the right hand side is in $W^{1,p}$.

Now we proceed to proof the openness of U_ϵ . In contrast to the previous lemma this is more delicate; we split the arguments into several lemmata.

Lemma 3.4. *There exists a constant $\kappa = \kappa(n) > 0$ such that for any $\zeta \in W^{2,p}(\mathbb{B}^n, so(m) \otimes \Lambda^{n-2}\mathbb{R}^n)$ with $\|d\zeta\|_{L^n} \leq \kappa$ there exists $\eta > 0$ such that for any $\lambda \in W^{1,p}(\mathbb{B}^n, so(m) \otimes \Lambda^1\mathbb{R}^n)$ with $\|\lambda\| \leq \eta$ the equation*

$$(3.5) \quad *d * (Q^{-1}dQ + Q^{-1}(*d\zeta + \lambda)Q) = 0$$

has a solution $Q = Q(\lambda) \in W^{2,p}(\mathbb{B}^n, SO(m))$.

Note that (3.5) implies, through Poincaré's lemma, that the term in parentheses is of the form $*d\tilde{\zeta}$, for some antisymmetric $(n-2)$ -form $\tilde{\zeta}$. The above Lemma should be understood as follows: Uhlenbeck's decomposition (i.e. Q and $\tilde{\zeta}$) exists if our matrix is a small (in $W^{1,p}$) perturbation of a co-closed form $*d\zeta$, provided $\|d\zeta\|_{L^n}$ is sufficiently small.

Proof. Since we are interested in finding any $Q \in W^{2,p}(\mathbb{B}^n, SO(m))$ satisfying (3.5), we shall look for one of the form e^u , where $u \in \mathcal{B} = W^{2,p}(\mathbb{B}^n, so(m)) \cap W_o^{1,p}(\mathbb{B}^n, so(m))$ (this adds boundary restrictions on Q). We define the operator

$$T : \mathcal{B} \times W^{1,p}(\mathbb{B}^n, so(m) \otimes \Lambda^1 \mathbb{R}^n) \rightarrow L^p(\mathbb{B}^n)$$

by

$$T(u, \lambda) = *d * (e^{-u} de^u + e^{-u} (*d\zeta + \lambda)e^u).$$

This is a well defined, smooth operator. Using Implicit Function Theorem, we prove that for any sufficiently small λ the equation $T(u, \lambda) = 0$ has a solution u_λ , continuously depending on λ . To this end we linearize T at $(u, \lambda) = (0, 0)$ with respect to the first argument:

$$H(\psi) = *d * (d\psi + [*d\zeta, \psi]) = \Delta\psi + [*d\zeta, d\psi], \quad H : \mathcal{B} \rightarrow L^p(\mathbb{B}^n),$$

where the commutator $[\cdot, \cdot]$ denotes a commutator of two $so(m)$ matrices.

We have, by Hölder's inequality

$$\begin{aligned} \|\Delta\psi\|_{L^p} &\leq \|H(\psi)\|_{L^p} + \|[d\zeta, d\psi]\|_{L^p} \\ &\leq \|H(\psi)\|_{L^p} + 2\|d\zeta\|_{L^n} \|d\psi\|_{L^{np/(n-p)}}, \end{aligned}$$

thus,

$$\begin{aligned} \|H(\psi)\|_{L^p} &\geq \|\Delta\psi\|_{L^p} - 2\|d\zeta\|_{L^n} \|d\psi\|_{L^{np/(n-p)}} \\ &\geq \|\Delta\psi\|_{L^p} - 2C_S \|d\zeta\|_{L^n} \|d\psi\|_{W^{1,p}} \\ &\geq C(\|d\psi\|_{W^{1,p}} - 2\kappa \|d\psi\|_{W^{1,p}}) \\ &= C(1 - 2\kappa) \|d\psi\|_{W^{1,p}}, \end{aligned}$$

where the constant C_S comes from the Sobolev embedding. Therefore, for κ small, H is injective.

Showing surjectivity of H amounts to showing that the system

$$(3.6) \quad H(\psi) = \Delta\psi + [*d\zeta, d\psi] = f$$

has a solution in \mathcal{B} for arbitrary $f \in L^p$.

Let us consider an operator $K : \mathcal{B} \rightarrow \mathcal{B}$, with $K(\psi)$ defined as a solution to the system

$$(3.7) \quad \begin{cases} \Delta K(\psi) = -[*d\zeta, d\psi] & \text{in } \mathbb{B}^n, \\ K(\psi) = 0 & \text{on } \partial\mathbb{B}^n. \end{cases}$$

Using Hölder's and Sobolev's inequalities and the fact that the Newtonian potential $\Delta^{-1} : L^p \rightarrow \mathcal{B}$ is continuous, we get

$$\begin{aligned} \|K(\psi)\|_{\mathcal{B}} &\leq \|\Delta^{-1}\|_{L^p \rightarrow \mathcal{B}} \|[*d\zeta, d\psi]\|_{L^p} \\ &\leq 2\|\Delta^{-1}\|_{L^p \rightarrow \mathcal{B}} \|d\zeta\|_{L^n} \|d\psi\|_{L^{np/(n-p)}} \\ &\leq 2\|\Delta^{-1}\|_{L^p \rightarrow \mathcal{B}} \kappa \|\psi\|_{\mathcal{B}}. \end{aligned}$$

For κ sufficiently small we can have $\|K\|_{\mathcal{B} \rightarrow \mathcal{B}}$ small and $Id - K : \mathcal{B} \rightarrow \mathcal{B}$ invertible.

Let now $\phi = (Id - K)\psi$. If ψ is a solution to (3.6), then

$$\Delta\phi = \Delta\psi - \Delta K(\psi) = \Delta\psi + *[d\zeta, d\psi] = f,$$

and we can solve (3.6) for any $f \in L^p$ by solving the above Poisson equation and applying to its solution the inverse mapping to $Id - K$.

Altogether, $H : \mathcal{B} \rightarrow L^p$ is an isomorphism, and we can apply the Implicit Function Theorem to get u_λ as a continuous function of λ .

To end the proof of the lemma, we take $Q(\lambda) = e^{u_\lambda}$. \square

Lemma 3.5. *Suppose $n/2 < p < n$. There exists $\kappa = \kappa(p, n)$ such that for any $\Omega \in V_\epsilon$ and P, ξ in $W^{2,p}$ satisfying the system (3.1) and additionally the estimate*

$$(3.8) \quad \|dP\|_{L^n} + \|d\xi\|_{L^n} < \kappa$$

the estimates (3.2) hold.

Proof. The lemma follows from rather standard elliptic estimates, but we include them here for the sake of completeness.

We have

$$(3.9) \quad \begin{aligned} \Delta\xi &= (*d * d + d * d*)\xi = *d(*d\xi) \\ &= *d(P^{-1}dP + P^{-1}\Omega P) \\ &= *(dP^{-1} \wedge dP) + *d(P^{-1}\Omega P). \end{aligned}$$

Note that for $q = p/(p-1)$, $\|d\xi\|_{L^p}$ is equivalent to

$$\sup_{\|d\phi\|_{L^q} \leq 1} \int_{\mathbb{B}^n} d\xi \cdot d\phi,$$

where ϕ is a smooth, compactly supported (in particular with null boundary values) $(n-2)$ -form on \mathbb{B}^n . The inequality

$$\|d\xi\|_{L^p} = \sup_{\|\eta\|_{L^q} \leq 1} \int_{\mathbb{B}^n} d\xi \cdot \eta \geq \sup_{\|d\phi\|_{L^q} \leq 1} \int_{\mathbb{B}^n} d\xi \cdot d\phi$$

is obvious. Applying the Hodge decomposition to the $(n-1)$ -form η : $\eta = d\phi + \psi$ with $\delta\psi = 0$, $\|d\phi\|_{L^q} \leq C_q \|\eta\|_{L^q}$, we get

$$\begin{aligned} \|d\xi\|_{L^p} &= \sup_{\|\eta\|_{L^q} \leq 1} \int_{\mathbb{B}^n} d\xi \cdot \eta \leq \sup_{\|\eta\|_{L^q} \leq 1} \left(\int_{\mathbb{B}^n} d\xi \cdot d\phi + \int_{\mathbb{B}^n} d\xi \cdot \psi \right) \\ &= C_q \sup_{\|\eta\|_{L^q} \leq 1} \left(\int_{\mathbb{B}^n} d\xi \cdot d\frac{\phi}{C_q} - \int_{\mathbb{B}^n} \xi \wedge \delta\psi \right) \leq C_q \sup_{\|d\phi\|_{L^q} \leq 1} \int_{\mathbb{B}^n} d\xi \cdot d\tilde{\phi}. \end{aligned}$$

Denote by \bar{P} the mean value of P over \mathbb{B}^n : $\bar{P} = \oint_{\mathbb{B}^n} P$. For any ϕ as above, with $\|d\phi\|_{L^q} \leq 1$,

$$\begin{aligned}
\int_{\mathbb{B}^n} d\xi \cdot d\phi &= - \int_{\mathbb{B}^n} \Delta\xi \cdot \phi \\
&= - \int_{\mathbb{B}^n} \left[* (dP^{-1} \wedge d(P - \bar{P})) + *d(P^{-1}\Omega P) \right] \cdot \phi \\
&= - \int_{\mathbb{B}^n} [dP^{-1}(P - \bar{P}) + P^{-1}\Omega P] \wedge d\phi \\
&\leq C \|dP^{-1}\|_{L^p} \|P - \bar{P}\|_{BMO} \|d\phi\|_{L^q} + \|P^{-1}\Omega P\|_{L^p} \|d\phi\|_{L^q} \\
&\leq C \|dP\|_{L^p} \|dP\|_{L^n} + \|\Omega\|_{L^p} \\
&\leq \kappa C \|dP\|_{L^p} + \|\Omega\|_{L^p},
\end{aligned}$$

thus

$$(3.10) \quad \|d\xi\|_{L^p} \leq C_q(\kappa C \|dP\|_{L^p} + \|\Omega\|_{L^p}),$$

with the constants C (possibly different in every line) dependent only on n and p . Note that, since P is an orthogonal matrix, $|P| = |P^{-1}| = 1$ and $|dP^{-1}| = |dP|$. In the estimate above we use, for the first summand, the Coifman-Lions-Meyer-Semmes div-curl inequality ([1]) and later the standard inclusion $W^{1,n} \hookrightarrow BMO$; the second summand is estimated by Hölder's inequality.

On the other hand, taking L^p norms of both sides of the equation

$$(3.11) \quad dP = P * d\xi + \Omega P$$

(c.f (3.1)) gives

$$(3.12) \quad \|dP\|_{L^p} \leq \|d\xi\|_{L^p} + \|\Omega\|_{L^p}$$

Putting (3.10) and (3.12) together we get

$$\begin{aligned}
\|d\xi\|_{L^p} + \|dP\|_{L^p} &\leq 2\|d\xi\|_{L^p} + \|\Omega\|_{L^p} \\
&\leq C_1(n, p)\kappa \|dP\|_{L^p} + C_2(n, p)\|\Omega\|_{L^p},
\end{aligned}$$

and for $\kappa < \frac{1}{C_1}$ this implies that the estimate (3.2a) holds.

The above calculation is valid also for $p = n$, which yields the estimate (3.2b).

To show the estimate (3.2c), by taking $*d*$ of both sides of (3.11), we see that

$$\begin{aligned}
\Delta P &= *d * dP = *d * (P * d\xi + \Omega P) \\
&= *(dP \wedge d\xi) + *d * (\Omega P),
\end{aligned}$$

thus

$$\begin{aligned}
(3.13) \quad \|dP\|_{W^{1,p}} &= \|dP\|_{L^p} + \|\Delta P\|_{L^p} \\
&\leq \|dP\|_{L^p} + \|dP \wedge d\xi\|_{L^p} + \|d\Omega P\|_{L^p} + \|\Omega \wedge dP\|_{L^p} \\
&\leq \|dP\|_{L^p} + \|dP\|_{L^n} \|d\xi\|_{L^{np/(n-p)}} + \|d\Omega\|_{L^p} + \|\Omega\|_{L^{np/(n-p)}} \|dP\|_{L^n} \\
&\leq \|dP\|_{L^p} + C\kappa \|d\xi\|_{W^{1,p}} + (1 + C\kappa)\|\Omega\|_{W^{1,p}}.
\end{aligned}$$

with the constant C coming from the Sobolev embedding $W^{1,p} \hookrightarrow L^{np/(n-p)}$, thus dependent only on p and n .

Similarly, using (3.9),

(3.14)

$$\begin{aligned} \|d\xi\|_{W^{1,p}} &= \|d\xi\|_{L^p} + \|\Delta\xi\|_{L^p} = \|d\xi\|_{L^p} + \|*(dP^{-1} \wedge dP) + *(P^{-1}\Omega P)\|_{L^p} \\ &\leq \|d\xi\|_{L^p} + \|dP\|_{L^n} \|dP\|_{L^{np/(n-p)}} + \|dP\|_{L^n} \|\Omega P\|_{L^{np/(n-p)}} \\ &\quad + \|P^{-1}d\Omega P\|_{L^p} + \|P^{-1}\Omega\|_{L^{np/(n-p)}} \|dP\|_{L^n} \\ &\leq \|d\xi\|_{L^p} + \kappa C \|dP\|_{W^{1,p}} + (2\kappa C + 1) \|\Omega\|_{W^{1,p}}. \end{aligned}$$

Composing (3.13) and (3.14) with the already proved estimate (3.2a) we get

$$\|dP\|_{W^{1,p}} + \|d\xi\|_{W^{1,p}} \leq \kappa C \|dP\|_{W^{1,p}} + \kappa C \|d\xi\|_{W^{1,p}} + (\kappa C + 1) \|\Omega\|_{W^{1,p}},$$

which, for κ sufficiently small, yields the estimate (3.2c). \square

Lemma 3.6. *The set U_ϵ is, for ϵ sufficiently small, open in V_ϵ .*

Proof. Choose $\Omega_o \in U_\epsilon$ and let P_o and ξ_o be the orthogonal transformation and antisymmetric $(n-2)$ -form associated with Ω_o , so that Theorem 3.1 holds for Ω_o , P_o and ξ_o .

Take now $\Omega \in V_\epsilon$ close to Ω_o in $W^{1,p}$: we ask that for $\lambda = P_o^{-1}(\Omega - \Omega_o)P_o$ we have $\|\lambda\|_{W^{1,p}} < \eta$ (the conjugation with $P_o \in W^{2,p}$ is continuous in $W^{1,p}$). Applying Lemma 3.4 with $\zeta = \xi_o$ we find $Q \in W^{2,p}(\mathbb{B}^n, SO(m))$ such that

$$(3.15) \quad *d*(Q^{-1}dQ + Q^{-1}(*d\xi_o + P_o^{-1}(\Omega - \Omega_o)P_o)Q) = 0.$$

Setting $P = P_o Q$ we see that (3.15) reduces to

$$*d*(P^{-1}dP + P^{-1}\Omega P) = 0.$$

By Poincaré's Lemma, this implies that $P^{-1}dP + P^{-1}\Omega P$ is a coexact form, i.e. there exists an antisymmetric $(n-2)$ -form ξ such that

$$(3.16) \quad *d\xi = P^{-1}dP + P^{-1}\Omega P,$$

thus P and ξ give Uhlenbeck's decomposition of Ω .

Note that Q and $P_o \in W^{2,p} \cap L^\infty$ imply that $P \in W^{2,p}$. By the Hodge decomposition theorem we can choose ξ to be coclosed ($d*\xi = 0$ on \mathbb{B}^n) and to have zero boundary values ($\xi|_{\partial\mathbb{B}^n} = 0$). Finally, the right hand side of (3.16) is in $W^{1,p}$, which gives $\xi \in W^{2,p}$.

What remains to prove is that P , ξ and Ω satisfy the estimates (3.2). Observe that if $\|\Omega - \Omega_o\|_{W^{1,p}}$ is small enough, then by continuity of the mapping $\lambda \mapsto u_\lambda$ so is $\|P - P_o\|_{W^{1,p}}$ and $\|\xi - \xi_o\|_{W^{1,p}}$; choosing η (measuring the distance $\|\Omega - \Omega_o\|_{W^{1,p}}$) sufficiently small we may have

$$\|P - P_o\|_{W^{1,p}} + \|\xi - \xi_o\|_{W^{1,p}} < \epsilon.$$

We also know that

$$\|d\xi_o\|_{L^n} + \|dP_o\|_{L^n} \leq C\|\Omega_o\|_{L^n} \leq C\epsilon$$

(this follows from $\Omega_o \in U_\epsilon$).

Taking ϵ sufficiently small, we may ensure that

$$\begin{aligned} \|d\xi\|_{L^n} + \|dP\|_{L^n} &\leq \|d\xi - d\xi_o\|_{L^n} + \|P - P_o\|_{L^n} + \|d\xi_o\|_{L^n} + \|dP_o\|_{L^n} \\ &< (C+1)\epsilon < \kappa, \end{aligned}$$

with κ as in Lemma 3.5. Applying this lemma we show that the estimates (3.2) hold.

Altogether, $\Omega \in U_\epsilon$, which proves the openness of U_ϵ . \square

Proof of Theorem 3.1. Since, by Lemmata 3.3 and 3.6, for ϵ sufficiently small the set U_ϵ is closed and open in V_ϵ , and since the latter is path connected (it is star-shaped in $W^{1,p}$), it follows that $U_\epsilon = V_\epsilon$, which proves Theorem 3.1. \square

Note that, for $\Omega \in W^{1,n/2}$, the proof of existence of the decomposition, i.e. Lemma 3.4, fails. However, we can proceed by a standard density argument, obtaining

Corollary 3.7. *Let $\Omega \in W^{1,n/2}$. There exists $\epsilon > 0$ such that if Ω is an antisymmetric matrix of 1-differential forms on \mathbb{B}^n such that*

$$\|\Omega\|_{L^n} < \epsilon$$

then there exist $P \in W^{2,n/2}(\mathbb{B}^n, SO(m))$ and $\xi \in W^{2,n/2}(\mathbb{B}^n, so(m) \otimes \Lambda^{n-2}\mathbb{R}^n)$ satisfying the system

$$\begin{cases} P^{-1}dP + P^{-1}\Omega P = *d\xi & \text{on } \mathbb{B}^n, \\ d*\xi = 0 & \text{on } \mathbb{B}^n, \\ \xi = 0 & \text{on } \partial\mathbb{B}^n; \end{cases}$$

and such that

$$\begin{aligned} \|d\xi\|_{W^{1,n/2}} + \|dP\|_{W^{1,n/2}} &\leq C(n, m)\|\Omega\|_{W^{1,n/2}}, \\ \|d\xi\|_{L^n} + \|dP\|_{L^n} &\leq C(n, m)\|\Omega\|_{L^n} < C\epsilon. \end{aligned}$$

4. UHLENBECK'S DECOMPOSITION, THE CASE $1 < p < n/2$

Theorem 4.1. *Assume $1 < p < n/2$. Let*

$$\Omega : \mathbb{B}^n \rightarrow so(m) \otimes \Lambda^1\mathbb{R}^n$$

be an antisymmetric matrix of 1-differential forms on \mathbb{B}^n . Assume

$$\Omega \in L^{2p,n-2p} \quad \text{and} \quad d\Omega \in L^{p,n-2p}$$

There exists $\epsilon > 0$ such that if Ω satisfies the smallness condition

$$\|\Omega\|_{L^{2p,n-2p}} < \epsilon$$

then there exist $P \in W^{2,p}(\mathbb{B}^n, SO(m))$ and $\xi \in W^{2,p}(\mathbb{B}^n, so(m) \otimes \Lambda^{n-2}\mathbb{R}^n)$ satisfying the system

$$(4.1) \quad \begin{cases} P^{-1}dP + P^{-1}\Omega P = *d\xi & \text{on } \mathbb{B}^n, \\ d*\xi = 0 & \text{on } \mathbb{B}^n, \\ \xi = 0 & \text{on } \partial\mathbb{B}^n. \end{cases}$$

Moreover $P, \xi \in L_2^{p,n-2p}$ with

$$(4.2a) \quad \|d\xi\|_{L^{p,n-p}} + \|dP\|_{L^{p,n-p}} \leq C(n, m)\|\Omega\|_{L^{2p,n-2p}}$$

$$(4.2b) \quad \|\Delta\xi\|_{L^{p,n-2p}} + \|\Delta P\|_{L^{p,n-2p}} \leq C(n, m)(\|\Omega\|_{L^{2p,n-2p}} + \|d\Omega\|_{L^{p,n-2p}}).$$

Remark. Observe that for $n = 2p$, by the Sobolev Embedding Theorem we have automatically $\Omega \in L^n$. The Morrey space $L^{2p,n-2p}$ equals L^n and the smallness condition for the norm of Ω agrees with the original, Sobolev space version of Uhlenbeck's theorem.

As previously, we shall break the proof of Theorem 4.1 into several lemmata. The proof of existence of P and ξ (Lemma 3.4) cannot be adapted to the present situation. To avoid this difficulty we first prove the Uhlenbeck result under more stringent regularity assumptions (see Lemma 4.2 below). To prove the existence of the elements of decomposition we follow the strategy of Tao and Tian [14]. At a certain moment of the proof (Lemma 4.4) we use the fact that due to the Morrey-Sobolev embedding (Proposition 2.3), for $\alpha > 0$,

$$L_1^{p,n-p+\alpha} \hookrightarrow C^0.$$

This is not true for $L_1^{p,n-p}$. Also, as pointed out in [16], C^0 is not dense in $L^{p,s}$.

Lemma 4.2. *Let $1 < p < n/2$. There exists $\epsilon > 0$ such that for every $\alpha > 0$ and for every*

$$\Omega \in L^{2p,n-2p+2\alpha} \quad \text{such that} \quad d\Omega \in L^{p,n-2p},$$

if Ω satisfies the smallness condition

$$\|\Omega\|_{L^{2p,n-2p}} < \epsilon$$

then there exist $P, \xi \in L_2^{p,n-2p+2\alpha}$ satisfying the system (4.1) and the estimates

$$(4.3a) \quad \|d\xi\|_{L^{p,n-p}} + \|dP\|_{L^{p,n-p}} \leq C(n, m)\|\Omega\|_{L^{2p,n-2p}} < C\epsilon,$$

$$(4.3b) \quad \|d\xi\|_{L^{p,n-p+\alpha}} + \|dP\|_{L^{p,n-p+\alpha}} \leq C(n, m)\|\Omega\|_{L^{2p,n-2p+2\alpha}},$$

$$(4.3c) \quad \|\Delta\xi\|_{L^{p,n-2p}} + \|\Delta P\|_{L^{p,n-2p}} \leq C(n, m) (\|\Omega\|_{L^{2p,n-2p}} + \|d\Omega\|_{L^{p,n-2p}}).$$

Proof of the Lemma 4.2. As in the Sobolev case, for $\alpha, \epsilon > 0$ we introduce sets

$$V_\epsilon^\alpha = \{\Omega \in L^{2p,n-2p+2\alpha} \cap L_1^{p,n-2p} : \|\Omega\|_{L^{2p,n-2p}} < \epsilon\}$$

$$U_\epsilon^\alpha = \{\Omega \in L^{2p,n-2p+2\alpha} \cap L_1^{p,n-2p} : \|\Omega\|_{L^{2p,n-2p}} < \epsilon$$

and there exist P and ξ satisfying the system (4.1) and estimates (4.3)}

In Lemmata 4.3 and 4.7 below we show that for ϵ sufficiently small the set U_ϵ^α is closed and open in V_ϵ^α , and since the latter is path connected (it is star-shaped), it follows that $U_\epsilon^\alpha = V_\epsilon^\alpha$. This (up to the proofs of these lemmata) completes the proof of the lemma. \square

Lemma 4.3. *The set U_ϵ^α is closed in V_ϵ^α .*

Proof. Suppose (Ω_k) is a sequence in U_ϵ^α convergent in $L_1^{p,n-2p}$ to some Ω . Observe that $L_1^{p,n-2p}$ embeds continuously in $W^{1,p}$. Therefore the sequence (Ω_k) is convergent in $W^{1,p}$.

With every Ω_k we have associated a pair P_k, ξ_k that satisfy (4.1):

$$\begin{cases} P_k^{-1}dP_k + P_k^{-1}\Omega_k P_k = *d\xi_k & \text{on } \mathbb{B}^n, \\ d*\xi_k = 0 & \text{on } \mathbb{B}^n, \\ \xi_k = 0 & \text{on } \partial\mathbb{B}^n. \end{cases}$$

We also have the estimates (4.3), in particular

$$\begin{aligned} \|d\xi\|_{L^{p,n-p+\alpha}} + \|dP\|_{L^{p,n-p+\alpha}} &\leq C(n, m)\|\Omega\|_{L^{2p,n-2p+2\alpha}} \\ \|\Delta\xi\|_{L^{p,n-2p}} + \|\Delta P\|_{L^{p,n-2p}} &\leq C(n, m)(\|\Omega\|_{L^{2p,n-2p}} + \|d\Omega\|_{L^{p,n-p}}). \end{aligned}$$

The inclusion of $L_1^{p,n-p+\alpha}$ in $W^{1,p}$, the boundary condition on ξ_k and boundedness of P_k ($|P_k| = 1$) allow us to interpret the above as boundedness of P_k and ξ_k in $W^{2,p}$. We can thus assume (after passing to subsequences) that P_k and ξ_k are weakly convergent in $W^{2,p}$ to some P and ξ .

Both the boundary condition $\xi|_{\partial\mathbb{B}} = 0$ and the condition $d*\xi = 0$ are conserved when passing to the weak limit. Moreover, since $n > 2p > p$, after passing to a subsequence,

$$\begin{aligned} \Omega_k \rightharpoonup \Omega & \quad \text{in } W^{1,p} & \Rightarrow & \quad \Omega_k \rightarrow \Omega & \quad \text{in } L^{\frac{np}{n-p}}, \\ P_k \rightharpoonup P & \quad \text{in } W^{2,p} & \Rightarrow & \quad dP_k \rightarrow dP & \quad \text{in } L^n, \\ \xi_k \rightharpoonup \xi & \quad \text{in } W^{2,p} & \Rightarrow & \quad d\xi_k \rightharpoonup d\xi & \quad \text{in } L^{\frac{np}{n-p}}, \end{aligned}$$

and for any small $\delta > 0$, Ω_k, dP_k and $d\xi_k$ converge strongly (to Ω, P and ξ , respectively) in L^s , for any $s < \frac{np}{n-p}$. We also know that P_k are uniformly bounded in L^∞ and strongly convergent, by Sobolev embedding theorem, in L^q for any q . This is enough to show the strong convergence of $\Omega_k P_k$ to ΩP in L^s ; altogether, we may pass to the strong limit in L^s in the system (3.3), showing that the equation

$$dP + \Omega P = P * d\xi \quad \text{in } \mathbb{B}^n$$

is satisfied in the sense of distributions.

The estimates (4.3) for P and ξ are then obvious. □

Lemma 4.4. *Let*

$$\begin{aligned} \zeta &: \mathbb{B}^n \rightarrow so(m) \otimes \Lambda^{n-2}\mathbb{R}^n \\ \lambda &: \mathbb{B}^n \rightarrow so(m) \otimes \Lambda^1\mathbb{R}^n \\ Q &: \mathbb{B}^n \rightarrow SO(m). \end{aligned}$$

Assume ζ belongs to the Morrey space $L_2^{p,n-2p+2\alpha}$. There exists a constant $\kappa = \kappa(p, n) > 0$ such that if the following smallness condition is satisfied ¹

$$\|d\zeta\|_{L^{2p,n-2p}} \leq \kappa$$

then there exists a constant $\eta > 0$ such that

$$\text{if } \lambda \in L^{2p,n-2p+2\alpha} \cap L_1^{p,n-2p+2\alpha} \quad \text{with} \quad \|\lambda\|_{L^{2p,n-2p+2\alpha}} + \|d\lambda\|_{L^{p,n-2p+2\alpha}} \leq \eta$$

then there exists a solution $Q \in L_2^{p,n-2p+2\alpha}$ of the equation

$$(4.4) \quad *d*(Q^{-1}dQ + Q^{-1}(*d\zeta + \lambda)Q) = 0.$$

¹the smallness condition on $d\zeta$ is sensible due to Proposition 2.4

Note that (4.4) implies, through Poincaré's lemma, that the term in parentheses is of the form $*d\tilde{\zeta}$, for some antisymmetric $(n-2)$ -form $\tilde{\zeta}$.

Proof. Since we are interested in finding any Q satisfying (4.4), we shall look for one of the form e^u , where

$$u: \mathbb{B}^n \rightarrow so(m), \quad u \in L_2^{p,n-2p+2\alpha}$$

This adds boundary restrictions on Q – since $Q(x) \in SO(m)$, we may assume it is close to identity at the boundary of \mathbb{B}^n , so u has zero boundary values.

The equation (4.4), together with the boundary condition, can be rewritten as

$$(4.5) \quad \begin{aligned} \Delta u &= *d * (du - e^{-u}de^u - e^{-u}(*d\zeta + \lambda)e^u) \\ u &= 0 \quad \text{on } \partial\mathbb{B}^n. \end{aligned}$$

We follow the proof of Tao and Tian [14], setting up the iteration scheme

$$(4.6) \quad \begin{aligned} \Delta u^{k+1} &= *d * F(u^k, \zeta, \lambda) \\ u^{k+1} &= 0 \quad \text{on } \partial\mathbb{B}^n \\ u^0 &= 0 \end{aligned}$$

where

$$(4.7) \quad F(u, \zeta, \lambda) = du - e^{-u}de^u - e^{-u}(*d\zeta + \lambda)e^u.$$

Some calculations need more subtle justification though, since we work in noncommutative setting. We will show that there exists $\delta > 0$ such that in each step of the induction

$$\text{if } \|u^k\|_{L_2^{p,n-2p+2\alpha}} \leq \delta \quad \text{then} \quad \|u^{k+1}\|_{L_2^{p,n-2p+2\alpha}} \leq \delta.$$

We start with an easy observation. Since

$$L_2^{p,n-2p+2\alpha} \hookrightarrow L_1^{p,n-p+\alpha} \hookrightarrow C^{0,\gamma} \quad \text{for some } \gamma$$

and Hölder-continuous functions on the bounded domain B are bounded, it follows that if $f \in L_1^{p,n-p+\alpha}(B)$ with $f = 0$ on $\partial\Omega$ then

$$(4.8) \quad \|f\|_\infty \leq [f]_{C^{0,\gamma}} \leq C_M \|f\|_{L_1^{p,n-p+\alpha}(B)} \leq C_M \|f\|_{L_2^{p,n-2p+2\alpha}(B)},$$

where C_M is the constant from the Morrey–Sobolev Embedding Lemma (see Proposition 2.3).

Therefore, for some $\beta \in (0, 1/2)$ to be fixed later, there exists $\delta = \delta(\beta)$, $\delta < \frac{\beta}{C_M}$, such that

$$(4.9) \quad \text{if } \|f\|_{L_2^{p,n-2p+2\alpha}(B)} < \delta \quad \text{then} \quad \|f\|_\infty \leq \beta.$$

Observe that, by Proposition 2.4,

$$(4.10) \quad \|d\zeta\|_{L_2^{p,n-2p+2\alpha}}^2 \leq \kappa \|\zeta\|_{L_2^{p,n-2p+2\alpha}},$$

Now we can start the induction, assuming

$$(4.11) \quad \|u^k\|_{L_2^{p,n-2p+2\alpha}(B)} < \delta < \text{an absolute constant (to be specified later)}.$$

We will show first that the following pointwise estimates hold

$$(4.12) \quad |*d*F(u^k, \zeta, \lambda)| \leq C(n) \left(|u^k| |D^2 u^k| + |du^k|^2 + (|d\zeta| + |\lambda|) |du^k| + |d\lambda| \right).$$

Indeed, applying the Campbell-Hausdorff-Baker formula we obtain

$$(4.13) \quad e^{-u} de^u = du - \frac{1}{2!} [u, du] + \frac{1}{3!} [u, [u, du]] - (\dots)$$

Therefore, taking the structure of the right hand side of (4.6) into account (see (4.7)), we obtain

$$\begin{aligned} & *d*F(u^k, \zeta, \lambda) \\ &= *d* \left(\sum_{j=2}^{\infty} \frac{(-1)^j}{j!} [u^k, [u^k, [u^k, \dots [u^k, du^k] \dots]] \right) - *d* (e^{-u^k} (*d\zeta + \lambda) e^{u^k}) \\ &= R_1 + R_2. \end{aligned}$$

We easily see that

$$\begin{aligned} |R_1| &\leq C(n) \sum_{j=2}^{\infty} \frac{2^j}{j!} \left(|D^2 u^k| |u^k|^{j-1} + (j-1) |du^k|^2 |u^k|^{j-2} \right) \\ |R_2| &\leq C(n) (|de^{u^k}| (|d\zeta| + |\lambda|) + |d\lambda|). \end{aligned}$$

Since (4.11) holds, we obtain

$$|R_1| \leq C(n) \left(|D^2 u^k| |u^k| + |du^k|^2 \right)$$

and

$$|R_2| \leq C(n) \left(|du^k| (|d\zeta| + |\lambda|) + |d\lambda| \right),$$

which proves the pointwise estimates (4.12) hold true, i.e.

$$|*d*F(u^k, \zeta, \lambda)| \leq C(n) \left(|u^k| |D^2 u^k| + |du^k|^2 + (|d\zeta| + |\lambda|) |du^k| + |d\lambda| \right).$$

Passing from pointwise to $L^{p, n-2p+2\alpha}$ estimates and using (4.10) we obtain

$$\begin{aligned} & \|*d*F(u^k, \zeta, \lambda)\|_{L^{p, n-2p+2\alpha}} \\ &\leq C_1(n, p) \left(\|u^k\|_{L^\infty} \|D^2 u^k\|_{L^{p, n-2p+2\alpha}} + \| |du^k|^2 \|_{L^{p, n-2p+2\alpha}} \right. \\ &\quad \left. + (\kappa \|\zeta\|_{L_2^{p, n-2p+2\alpha}} + \eta^2)^{1/2} \|u^k\|_{L_2^{p, n-2p+2\alpha}} + \eta \right) \\ &\leq C_2(n, p, C_M) \left(\|u^k\|_{L_2^{p, n-2p+2\alpha}}^2 + (\kappa \|\zeta\|_{L_2^{p, n-2p+2\alpha}} + \eta^2)^{1/2} \|u^k\|_{L_2^{p, n-2p+2\alpha}} + \eta \right), \end{aligned}$$

where the last inequality follows from (4.8), with u^k in place of f . Regularity estimates for linear elliptic systems yield

$$\begin{aligned} & \|u^{k+1}\|_{L_2^{p, n-2p+2\alpha}} \\ &\leq C_3(n, p, C_M) \left(\|u^k\|_{L_2^{p, n-2p+2\alpha}}^2 + (\kappa \|\zeta\|_{L_2^{p, n-2p+2\alpha}} + \eta^2)^{1/2} \|u^k\|_{L_2^{p, n-2p+2\alpha}} + \eta \right) \end{aligned}$$

W.l.o.g. we may assume $C_3 > 1$. Let us choose η and κ such that

$$\eta < \frac{1}{8C_3},$$

and

$$\kappa \|\zeta\|_{L_2^{p,n-2p+2\alpha}} < \frac{3}{64C_3^2}.$$

Set

$$\delta = 2C_3\eta.$$

Then, if

$$\|u^k\|_{L_2^{p,n-2p+2\alpha}} \leq \delta,$$

we have

$$\|u^{k+1}\|_{L_2^{p,n-2p+2\alpha}} \leq C_3 \left(\delta^2 + (\kappa \|\zeta\|_{L_2^{p,n-2p+2\alpha}} + \eta^2) \delta + \eta \right) \leq \delta.$$

Now, let us apply the same scheme to differences of u^k .

Fix a constant $\beta \in (0, 1/2)$ (it will be specified at the end). We adjust δ if necessary, i.e. we take

$$(4.14) \quad \delta < \min\{2C_3\eta; \frac{\beta}{C_M}\}$$

where C_M is the constant in (4.8). Then, we assume that $u, v \in L_2^{p,n-2p+2\alpha}$, $\|u\|_{L_2^{p,n-2p+2\alpha}} < \delta$, $\|v\|_{L_2^{p,n-2p+2\alpha}} < \delta$ and $u, v = 0$ on $\partial\mathbb{B}^n$. This, in particular, implies (by (4.9)), that $\|u\|_{L^\infty}$ and $\|v\|_{L^\infty}$ are at most β .

We have

$$\begin{aligned} F(u, \zeta, \lambda) - F(v, \zeta, \lambda) &= (d(u-v) - e^{-u}de^u + e^{-v}de^v) + (-e^{-u}(*d\zeta + \lambda)e^u + e^{-v}(*d\zeta + \lambda)e^v) \\ &= S_1 + S_2. \end{aligned}$$

By (4.13), we have

$$\begin{aligned} S_1 &= d(u-v) \\ &\quad - \left(du - \frac{1}{2!}[u, du] + \frac{1}{3!}[u, [u, du]] + \cdots + \frac{(-1)^{k-1}}{k!}[u, \dots, [u, [u, du]] \dots] + \cdots \right) \\ &\quad + \left(dv - \frac{1}{2!}[v, dv] + \frac{1}{3!}[v, [v, dv]] + \cdots + \frac{(-1)^{k-1}}{k!}[v, \dots, [v, [v, dv]] \dots] + \cdots \right) \\ &= \frac{1}{2!}([u-v, du] + [v, d(u-v)]) \\ &\quad - \frac{1}{3!}([u-v, [u, du]] + [v, [u-v, du]] + [v, [v, d(u-v)]]) + \cdots \\ &\quad + \frac{(-1)^k}{k!}([u-v, [u, \dots [u, du] \dots]] + [v, [u-v, [u, \dots [u, du] \dots]]) \\ &\quad + \cdots + [v, \dots [v, [u-v, du]] \dots] + [v, \dots [v, d(u-v)] \dots]). \end{aligned}$$

Likewise, S_2 can be rewritten as

$$\begin{aligned} S_2 &= -e^{-u}(*d\zeta + \lambda)e^u + e^{-v}(*d\zeta + \lambda)e^v \\ &= -(e^{-u} - e^{-v})(*d\zeta + \lambda)e^u - e^{-v}(*d\zeta + \lambda)(e^u - e^v). \end{aligned}$$

Next, we want to estimate (pointwise) $*d * S_1$. Keeping in mind that an ℓ -tuple commutator $[a_1, [a_2, \dots [a_{\ell-1}, a_\ell] \dots]]$ stands for a sum of $2^{\ell-1}$ products of the form $a_{i_1} a_{i_2} \dots a_{i_\ell}$, we obtain

$$\begin{aligned}
|*d * S_1| \leq & \sum_{\ell=2}^{\infty} \frac{2^{\ell-1}}{\ell!} \left(|d(u-v)||u|^{\ell-2}|du| \right. \\
& + (\ell-2)|u-v||u|^{\ell-3}|du|^2 + |u-v||u|^{\ell-2}|D^2u| \\
& + \sum_{j=1}^{\ell-3} \left(j|dv||v|^{j-1}|u-v||u|^{\ell-j-2}|du| + |v|^j|d(u-v)||u|^{\ell-j-2}|du| \right. \\
& \quad \left. + (\ell-j-2)|v|^j|u-v||u|^{\ell-j-3}|du|^2 + |v|^j|u-v||u|^{\ell-j-2}|D^2u| \right) \\
& + (\ell-2)|v|^{\ell-3}|dv||u-v||du| + |v|^{\ell-2}|d(u-v)||du| + |v|^{\ell-2}|u-v||D^2u| \\
& \left. + (\ell-1)|v|^{\ell-2}|dv||u-v||du| + |v|^{\ell-1}|D^2(u-v)| \right).
\end{aligned}$$

Using the fact that $|u|, |v| < \beta$ and $2\beta < 1$, we estimate further

$$\begin{aligned}
(4.15) \quad |*d * S_1| \leq & |d(u-v)||du|\beta \sum_{\ell=2}^{\infty} \frac{4(\ell-1)}{\ell!} \\
& + |u-v||du|^2 \sum_{\ell=2}^{\infty} \frac{2(\ell-1)(\ell-2)}{\ell!} \\
& + |u-v||D^2u|\beta \sum_{\ell=2}^{\infty} \frac{4(\ell-1)}{\ell!} \\
& + |du||dv||u-v| \sum_{\ell=2}^{\infty} \frac{2}{\ell!} \\
& + |D^2(u-v)| \sum_{\ell=2}^{\infty} \frac{(2\beta)^{\ell-1}}{\ell!} \\
\leq & C\beta \left(|d(u-v)||du| + |u-v||D^2u| + |D^2(u-v)| \right) \\
& + C|u-v| \left(|du|^2 + |du||dv| \right),
\end{aligned}$$

where C is a universal constant.

To estimate $|*d * S_2|$, note that

$$\begin{aligned}
|e^u - e^v| & \leq \sum_{\ell=1}^{\infty} \frac{1}{\ell!} |(u)^\ell - (v)^\ell| \\
& = \sum_{\ell=1}^{\infty} \frac{1}{\ell!} \left| (u-v)(u)^{\ell-1} + v(u-v)(u)^{\ell-2} + \dots + (v)^{\ell-2}(u-v)u + (v)^{\ell-1}(u-v) \right| \\
& \leq |u-v| \sum_{\ell=1}^{\infty} \frac{\ell}{2^{\ell-1}\ell!} = \sqrt{e}|u-v|.
\end{aligned}$$

and

$$\begin{aligned} |d(e^u - e^v)| &= |e^u du - e^v dv| \leq |e^u - e^v| |du| + e^{|v|} |d(u - v)| \\ &\leq \sqrt{e} |u - v| |du| + \sqrt{e} |d(u - v)|. \end{aligned}$$

Thus

$$\begin{aligned} |*d*S_2| &\leq |d(e^{-u} - e^{-v})|(|d\zeta| + |\lambda|)|e^u| \\ &\quad + |e^{-u} - e^{-v}| |d\lambda| |e^u| + |e^{-u} - e^{-v}|(|d\zeta| + |\lambda|)|e^u du| \\ &\quad + |e^{-v} dv|(|d\zeta| + |\lambda|)|e^u - e^v| + |e^{-v}| |d\lambda| |e^u - e^v| \\ &\quad + |e^{-v}|(|d\zeta| + |\lambda|)|d(e^u - e^v)| \\ (4.16) \quad &\leq C|u - v|(|d\lambda| + (|du| + |dv|)(|d\zeta| + |\lambda|)) \\ &\quad + C|d(u - v)|(|d\zeta| + |\lambda|). \end{aligned}$$

Altogether, adding up the estimates (4.15) and (4.16), we obtain the following pointwise estimate

$$\begin{aligned} (4.17) \quad &|*d*(F(u, \zeta, \lambda) - F(v, \zeta, \lambda))| \\ &\leq C|u - v|(|du|^2 + |D^2 u| + |du||dv| + |d\lambda| + (|du| + |dv|)(|d\zeta| + |\lambda|)) \\ &\quad + C\beta(|d(u - v)||du| + |D^2(u - v)|) \\ &\quad + C|d\zeta||d(u - v)| \\ &\quad + C|\lambda||d(u - v)|, \end{aligned}$$

with C a universal constant.

Passing from the pointwise to $L^{p, n-2p+2\alpha}$ estimates, using (4.8), (4.10) and (2.1) and keeping in mind all smallness conditions, i.e.

$$\begin{aligned} \|d\zeta\|_{L^{2p, n-2p+2\alpha}}^2 &\leq \kappa \|\zeta\|_{L_2^{p, n-2p+2\alpha}}, \\ \|\lambda\|_{L^{2p, n-2p+2\alpha}} + \|d\lambda\|_{L^{p, n-2p+2\alpha}} &\leq \eta, \\ \|u\|_{L_2^{p, n-2p+2\alpha}} &\leq \delta, \\ \|v\|_{L_2^{p, n-2p+2\alpha}} &\leq \delta, \\ \|u\|_{L^\infty} &< \beta \\ \|v\|_{L^\infty} &< \beta \end{aligned}$$

we obtain

$$\begin{aligned} &\|*d*(F(u, \zeta, \lambda) - F(v, \zeta, \lambda))\|_{L_2^{p, n-2p+2\alpha}} \\ &\leq C\|u - v\|_\infty (\delta^2 + \delta + \eta + \delta\kappa^{1/2} + \delta\eta) + \|u - v\|_{L_2^{p, n-2p+2\alpha}} (\beta + \kappa^{1/2} + \eta) \\ &\leq C\|u - v\|_{L_2^{p, n-2p+2\alpha}} (\delta^2 + \delta + \eta + \delta\kappa^{1/2} + \delta\eta + \beta + \kappa^{1/2} + \eta) \end{aligned}$$

If we denote $H(u^k) = u^{k+1}$, where u^{k+1} is a solution to (4.6), then

$$\begin{aligned} &\|H(u) - H(v)\|_{L_2^{p, n-2p+2\alpha}} \\ &\leq C_E C \|u - v\|_{L_2^{p, n-2p+2\alpha}} (\delta^2 + \delta + \eta + \delta\kappa^{1/2} + \delta\eta + \beta + \kappa^{1/2} + \eta), \end{aligned}$$

where C_E is an absolute constant from elliptic estimates (see [3]). Now, in order to show that H is a contraction, we choose β and κ and η sufficiently small. The choice of β and η results in the choice of δ . Therefore, by the Banach fixed point theorem, the iteration scheme (4.6) converges and we obtain the desired solution of the system (4.5). \square

Lemma 4.5. *Suppose $p < n/2$. There exists $\kappa = \kappa(p, n)$ with the following property: suppose that for $\Omega \in V_\epsilon^\alpha$ there exist P and ξ in $L_2^{p, n-2p+2\alpha}(\mathbb{B}^n)$ satisfying the system (4.1) and additionally the estimate*

$$(4.18) \quad \|dP\|_{L^{2p, n-2p}(\mathbb{B}^n)} + \|d\xi\|_{L^{2p, n-2p}(\mathbb{B}^n)} < \kappa.$$

Then the estimates (4.3) hold.

Remark 4.6. Note that the fact that $dP, d\xi \in L^{2p, n-2p}$ follows from Proposition 2.4.

Proof of Lemma 4.5. We have in the ball \mathbb{B}^n

$$(4.19) \quad \begin{aligned} \Delta\xi &= (*d * d + d * d*)\xi = *d(*d\xi) \\ &= *d(P^{-1}dP + P^{-1}\Omega P) \\ &= *(dP^{-1} \wedge dP) + *d(P^{-1}\Omega P). \end{aligned}$$

Let $B = B_r(x_0)$ be a fixed ball. On the set $B \cap \mathbb{B}^n$ we split ξ into a sum of two functions $\xi = u + v$, satisfying

$$\begin{aligned} \Delta u &= *(dP^{-1} \wedge dP) \quad \text{on } B \cap \mathbb{B}^n \\ u &= 0 \quad \text{on } \partial(B \cap \mathbb{B}^n) \end{aligned}$$

(we may assume $\text{supp } u \subset B \cap \mathbb{B}^n$) and

$$\begin{aligned} \Delta v &= *d(P^{-1}\Omega P) \quad \text{on } \mathbb{B}^n \\ v &= 0 \quad \text{on } \partial\mathbb{B}^n. \end{aligned}$$

Thus $\xi = u + v$ on $B \cap \mathbb{B}^n$.

For $q = p/(p-1)$ we have

$$\|du\|_{L^p(B \cap \mathbb{B}^n)} \leq C_q \sup_{\|d\phi\|_{L^q} \leq 1} \int_{B \cap \mathbb{B}^n} du \cdot d\phi,$$

where ϕ is a smooth, compactly supported (in particular with null boundary values) $(n-2)$ -form on $B \cap \mathbb{B}^n$ (c.f. the proof of Lemma 3.5). Denote by \bar{P} the mean value of P over \mathbb{B}^n : $\bar{P} = \oint_{\mathbb{B}^n} P$. For any ϕ as above,

$$\begin{aligned} \int_{B \cap \mathbb{B}^n} du \cdot d\phi &= - \int_{B \cap \mathbb{B}^n} \Delta u \cdot \phi \\ &= - \int_{\mathbb{B}^n} *(dP^{-1} \wedge d(P - \bar{P})) \cdot \phi \\ &= - \int_{\mathbb{B}^n} dP^{-1}(P - \bar{P}) \wedge d\phi \\ &\leq C \|dP^{-1} \wedge d\phi\|_{\mathcal{H}^1(\mathbb{B}^n)} \|P - \bar{P}\|_{BMO(\mathbb{B}^n)} \\ &= C \|dP^{-1} \wedge d\phi\|_{\mathcal{H}^1(B \cap \mathbb{B}^n)} \|P - \bar{P}\|_{BMO(\mathbb{B}^n)} \\ &\leq C \|dP\|_{L^p(B \cap \mathbb{B}^n)} \|d\phi\|_{L^q(B \cap \mathbb{B}^n)} \|dP\|_{L^{p, n-p}(\mathbb{B}^n)} \end{aligned}$$

with the constants C (possibly different in every line) dependent only on n and p . Note that since P is an orthogonal matrix, $|P| = |P^{-1}| = 1$, $|dP^{-1}| = |dP|$. In the estimate above we use the Coifman-Lions-Meyer-Semmes div-curl inequality ([1]) and later the inclusion

$$L_1^{p,n-p}(\mathbb{B}^n) \hookrightarrow BMO(\mathbb{B}^n).$$

We have then

$$\begin{aligned} \|du\|_{L^p(B \cap \mathbb{B}^n)} &\leq C \|dP\|_{L^p(B \cap \mathbb{B}^n)} \|dP\|_{L^{p,n-p}(\mathbb{B}^n)} \\ &\leq C \|dP\|_{L^p(B \cap \mathbb{B}^n)} \|dP\|_{L^{2p,n-2p}(\mathbb{B}^n)} \\ &\leq C \kappa \|dP\|_{L^p(B \cap \mathbb{B}^n)} \end{aligned}$$

due to the smallness assumption (4.18). Therefore

$$(4.20) \quad \|du\|_{L^{p,n-p+\alpha}(\mathbb{B}^n)} \leq C \kappa \|dP\|_{L^{p,n-p+\alpha}(\mathbb{B}^n)}$$

for $\alpha \geq 0$.

To estimate v , where

$$\begin{aligned} \Delta v &= *d(P^{-1}\Omega P) \quad \text{on } B \cap \mathbb{B}^n \\ v &= 0 \quad \text{on } (\partial \mathbb{B}^n) \cap B \end{aligned}$$

we use standard elliptic estimates obtaining

$$(4.21) \quad \|dv\|_{L^{p,n-p+\alpha}(\mathbb{B}^n)} \leq C \|\Omega\|_{L^{p,n-p+\alpha}(\mathbb{B}^n)}.$$

Combining (4.20) and (4.21) we conclude

$$(4.22) \quad \|d\xi\|_{L^{p,n-p+\alpha}(\mathbb{B}^n)} \leq C \kappa \|dP\|_{L^{p,n-p+\alpha}(\mathbb{B}^n)} + C \|\Omega\|_{L^{p,n-p+\alpha}(\mathbb{B}^n)}.$$

On the other hand, taking L^p norms of both sides of the equation

$$(4.23) \quad dP = P * d\xi + \Omega P \quad \text{in } \mathbb{B}^n$$

(c.f (4.1)) gives

$$\|dP\|_{L^p(B \cap \mathbb{B}^n)} \leq \|d\xi\|_{L^p(B \cap \mathbb{B}^n)} + \|\Omega\|_{L^p(B \cap \mathbb{B}^n)}$$

and thus

$$(4.24) \quad \|dP\|_{L^{p,n-p+\alpha}(\mathbb{B}^n)} \leq \|d\xi\|_{L^{p,n-p+\alpha}(\mathbb{B}^n)} + \|\Omega\|_{L^{p,n-p+\alpha}(\mathbb{B}^n)}.$$

Putting (4.22) and (4.24) together we get

$$\|d\xi\|_{L^{p,n-p+\alpha}(\mathbb{B}^n)} + \|dP\|_{L^{p,n-p+\alpha}(\mathbb{B}^n)} \leq C \kappa \|dP\|_{L^{p,n-p+\alpha}(\mathbb{B}^n)} + C \|\Omega\|_{L^{p,n-p+\alpha}(\mathbb{B}^n)},$$

for $\alpha \geq 0$ and taking κ small enough we conclude the estimates (4.3a) (4.3b) hold, i.e.

$$\|d\xi\|_{L^{p,n-p+\alpha}(\mathbb{B}^n)} + \|dP\|_{L^{p,n-p+\alpha}(\mathbb{B}^n)} \leq C \|\Omega\|_{L^{2p,n-2p+2\alpha}(\mathbb{B}^n)}$$

for $\alpha \geq 0$.

To show the estimate (4.3c), by taking $*d*$ of both sides of (4.23), we see that

$$\begin{aligned} \Delta P &= *d * dP = *d * (P * d\xi + \Omega P) \\ &= *(dP \wedge d\xi) + *d * (\Omega P), \end{aligned}$$

and from (4.19) we obtain

$$\Delta \xi = *(dP^{-1} \wedge dP) + *d(P^{-1}\Omega P).$$

Therefore, proceeding as in the proof of the estimates (3.13) and (3.14), we obtain

$$\|\Delta P\|_{L^{p,n-2p}} \leq \|dP\|_{L^{2p,n-2p}} \|d\xi\|_{L^{2p,n-2p}} + \|d\Omega\|_{L^{p,n-2p}} + \|\Omega\|_{L^{2p,n-2p}} \|dP\|_{L^{2p,n-2p}}$$

and

$$\|\Delta\xi\|_{L^{p,n-2p}} \leq \|dP\|_{L^{2p,n-2p}}^2 + 2\|dP\|_{L^{2p,n-2p}} \|\Omega\|_{L^{2p,n-2p}} + \|d\Omega\|_{L^{p,n-2p}}$$

Applying (4.18), we obtain

$$\|\Delta P\|_{L^{p,n-2p}} + \|\Delta\xi\|_{L^{p,n-2p}} \leq 2\|d\Omega\|_{L^{p,n-2p}} + 3\kappa\|\Omega\|_{L^{2p,n-2p}}$$

which proves the estimate (4.3c).

Observe that the above inequality is a consequence of the equation (4.23) and the Hölder inequality only, so the estimate holds in any Morrey space $L^{p,\gamma}$ with $\gamma > n - 2p$ in which both sides of the inequality are finite. \square

Lemma 4.7. *The set U_ϵ^α is, for ϵ sufficiently small, open in V_ϵ^α .*

Proof. Choose $\Omega_o \in U_\epsilon^\alpha$ and let P_o and ξ_o be the orthogonal transformation and antisymmetric $(n-2)$ -form associated with Ω_o , so that Lemma 4.2 holds for Ω_o , P_o and ξ_o .

Take now $\Omega \in V_\epsilon^\alpha$ close to Ω_o in $L^{2p,n-2p+2\alpha} \cap L_1^{p,n-2p+2\alpha}$: we ask that for $\lambda = P_o^{-1}(\Omega - \Omega_o)P_o$ we have

$$\|\lambda\|_{L^{2p,n-2p+2\alpha}} + \|\lambda\|_{L_1^{p,n-2p+2\alpha}} < \eta$$

(the conjugation with $P_o \in L_2^{p,n-2p+2\alpha}$ is continuous in $L_1^{p,n-2p+2\alpha}$). Applying Lemma 4.4 with $\zeta = \xi_o$ we find $Q \in L_2^{p,n-2p+2\alpha}(\mathbb{B}^n, SO(m))$ such that

$$(4.25) \quad *d*(Q^{-1}dQ + Q^{-1}(*d\xi_o + P_o^{-1}(\Omega - \Omega_o)P_o)Q) = 0.$$

Setting $P = P_oQ$ we see that (4.25) reduces to

$$*d*(P^{-1}dP + P^{-1}\Omega P) = 0.$$

By Poincaré's lemma this implies that $P^{-1}dP + P^{-1}\Omega P$ is a coexact form, i.e. there exists an antisymmetric $(n-2)$ -form ξ such that

$$(4.26) \quad *d\xi = P^{-1}dP + P^{-1}\Omega P,$$

thus P and ξ give Uhlenbeck's decomposition of Ω .

Note that Q and $P_o \in L_2^{p,n-2p+2\alpha} \cap L^\infty$ imply that $P \in L_2^{p,n-2p+2\alpha}$. By Hodge decomposition theorem we can choose ξ to be coclosed ($d*\xi = 0$ on \mathbb{B}^n) and to have zero boundary values ($\xi|_{\partial\mathbb{B}^n} = 0$). Finally, the right hand side of (4.26) is in $L_1^{p,n-2p+2\alpha}$, which gives $\xi \in L_2^{p,n-2p+2\alpha}$.

What remains to prove is that P , ξ and Ω satisfy the estimates (4.3). Observe that if $\|\Omega - \Omega_o\|_{L_1^{p,n-2p+2\alpha}}$ is small enough, then by continuity of the mapping $\lambda \mapsto u_\lambda$ so is $\|P - P_o\|_{L_1^{p,n-2p+2\alpha}}$ and $\|\xi - \xi_o\|_{L_1^{p,n-2p+2\alpha}}$; choosing η (which measures the distance $\|\Omega - \Omega_o\|_{L_1^{p,n-2p+2\alpha}}$) sufficiently small we get

$$\|P - P_o\|_{L_1^{p,n-2p+2\alpha}} + \|\xi - \xi_o\|_{L_1^{p,n-2p+2\alpha}} < \epsilon.$$

We also know that

$$\|d\xi_o\|_{L^{2p,n-2p}} + \|dP_o\|_{L^{2p,n-2p}} \leq C\|\Omega_o\|_{L^{2p,n-2p}} \leq C\epsilon$$

(this follows from $\Omega_o \in U_\epsilon$).

Taking ϵ sufficiently small we may ensure that

$$\begin{aligned} & \|d\xi\|_{L^{2p,n-2p}} + \|dP\|_{L^{2p,n-2p}} \\ & \leq \|d\xi - d\xi_o\|_{L^{2p,n-2p}} + \|P - P_o\|_{L^{2p,n-2p}} + \|d\xi_o\|_{L^{2p,n-2p}} + \|dP_o\|_{L^{2p,n-2p}} \\ & < (C+1)\epsilon < \kappa, \end{aligned}$$

with κ as in Lemma 4.5. Applying this lemma we show that the estimates (4.3) hold.

Altogether, $\Omega \in U_\epsilon^\alpha$, which proves the openness of U_ϵ^α . \square

Proof of Theorem 4.1. The proof mimics, in a way, the passage from Theorem 3.1 to Corollary 3.7, i.e. from the Uhlenbeck decomposition in $W^{1,p}$ for $p > n/2$ to the decomposition for $p = n/2$. There we could simply argue by a density argument. In the Morrey space setting, however, neither $L_1^{p,n-p+\alpha}$ embeds densely in $L_1^{p,n-p}$, nor $L^{2p,n-2p+2\alpha}$ does into $L^{2p,n-2p}$.

However (cf. [11], proof of Lemma 3.1), one can easily prove that if (ϕ_r) is a standard mollifier and $f \in L^{q,n-q}(\mathbb{B})$, $q \geq 1$, then on any ball $B = B(x, \rho)$ such that $2B = B(x, 2\rho) \subset \mathbb{B}^n$ we have, for $r < \rho$, that $\|f * \phi_r\|_{L^{q,n-q}(B)} \leq \|f\|_{L^{q,n-q}(2B)}$. Reasoning like in the proof of Meyers-Serrin's theorem and using a suitable decomposition of unity we can show then that there exists a sequence $f_k \in C^\infty(\mathbb{B})$ convergent to f in L^q (and in any other appropriate Lebesgue and Sobolev norm) such that $\|f_k\|_{L^{q,n-q}(\mathbb{B})} \leq C(n)\|f\|_{L^{q,n-q}(\mathbb{B})}$.

We thus proceed as follows: we approximate Ω in $W^{1,p}$ by a sequence of smooth Ω_k such that for all k

$$(4.27) \quad \|\Omega_k\|_{L^{2p,n-2p}(\mathbb{B})} \leq C(n)\|\Omega\|_{L^{2p,n-2p}(\mathbb{B})}.$$

Assuming that ϵ in the condition $\|\Omega\|_{L^{2p,n-2p}} < \epsilon$ is taken small enough we can ensure, through (4.27), that all Ω_k satisfy the analogous smallness condition in Lemma 4.2. This provides us with sequences of P_k and ξ_k that give the Uhlenbeck decomposition for Ω_k , together with the uniform estimate

$$\|d\xi_k\|_{L^{p,n-p}} + \|dP_k\|_{L^{p,n-p}} \leq C(n, m)\|\Omega\|_{L^{2p,n-2p}} < C\epsilon.$$

Then we proceed as in the proof of closedness of U_ϵ^α in Lemma 4.3, obtaining convergent subsequences of P_k and ξ_k . As Ω_k are smooth, they satisfy the assumptions of Lemma 4.2, which gives us the estimates (4.3)

$$\begin{aligned} & \|d\xi_k\|_{L^{p,n-p}} + \|dP_k\|_{L^{p,n-p}} \leq C(n, m)\|\Omega_k\|_{L^{2p,n-2p}}, \\ & \|\Delta\xi_k\|_{L^{p,n-2p}} + \|\Delta P_k\|_{L^{p,n-2p}} \leq C(n, m)(\|\Omega_k\|_{L^{2p,n-2p}} + \|\nabla\Omega_k\|_{L^{p,n-2p}}). \end{aligned}$$

The sequences P_k and ξ_k converge in $L_2^{p,n-2p}$ to appropriate elements of a decomposition of Ω (this follows from the equation they satisfy). Thus, the above estimates, together with (4.27), yield the desired estimates (4.3) for Ω . \square

5. UHLENBECK'S DECOMPOSITION AND CONFORMAL MATRICES

A natural extension of the orthogonal gauge group $SO(m)$ is the conformal group $CO_+(m)$. The interest in this group has deep roots in complex analysis, in particular in the study related to Liouville Theorem (see [6] for a detailed exposition). This is a non-compact group, defined as

$$CO_+(m) = \{\lambda P : \lambda \in \mathbb{R}_+, P \in SO(m)\}.$$

Clearly, $S \in CO_+(m)$ iff $SS^T = \lambda^2 \mathbb{I}$, where by \mathbb{I} we denote the $m \times m$ identity matrix.

The tangent space at \mathbb{I} to $CO_+(m)$, which we denote by $TCO_+(m)$, is given as

$$TCO_+ = \{K \in \mathcal{M}^{m \times m} : K + K^T = \frac{2Tr(K)}{m} \otimes \mathbb{I}\},$$

or, equivalently,

$$TCO_+ = \{A + \mu \mathbb{I} : A \in so(m), \mu \in \mathbb{R}\},$$

see e.g. [2].

Our objective is the following analogue of Theorem 3.1:

Theorem 5.1. *Let $\frac{n}{2} < p < n$. There exists $\epsilon > 0$ such that for any $\Omega \in W^{1,p}(\mathbb{B}^n, TCO_+(m) \otimes \Lambda^1 \mathbb{R}^n)$ such that $\|\Omega\|_{L^n} < \epsilon$ there exist $S : \mathbb{B}^n \rightarrow CO_+(m)$ satisfying $\ln |S| \in W^{2,p}(\mathbb{B}^n)$, $S/|S| \in W^{2,p}(\mathbb{B}^n, SO(m))$ and $\zeta \in W^{2,p}(\mathbb{B}^n, TCO_+(m) \otimes \Lambda^{n-2} \mathbb{R}^n)$ such that*

$$(5.1) \quad \begin{cases} S^{-1}dS + S^{-1}\Omega S = *d\zeta & \text{on } \mathbb{B}^n, \\ d*\zeta = 0 & \text{on } \mathbb{B}^n, \\ \zeta = 0 & \text{on } \partial \mathbb{B}^n; \end{cases}$$

and such that

$$(5.2a) \quad \|d\zeta\|_{W^{1,p}} + \|d(S/|S|)\|_{W^{1,p}} + \|d \ln |S|\|_{W^{1,p}} \leq C(n, m) \|\Omega\|_{W^{1,p}}$$

$$(5.2b) \quad \|d\zeta\|_{L^p} + \|d(S/|S|)\|_{L^p} + \|d \ln |S|\|_{L^p} \leq C(n, m) \|\Omega\|_{L^p},$$

$$(5.2c) \quad \|d\zeta\|_{L^n} + \|d(S/|S|)\|_{L^n} + \|d \ln |S|\|_{L^n} \leq C(n, m) \|\Omega\|_{L^n}.$$

The integrability conditions on $\ln |S|$ should be understood as (rather weak) integrability conditions both on S and S^{-1} . We should also note that if S satisfies the above theorem, so does tS for any non-zero constant t .

Proof. We shall construct $S : \mathbb{B}^n \rightarrow CO_+(m)$ satisfying the above conditions. Let us first fix some notation:

We shall write $S = \lambda P$, where $\lambda = |S| \in \mathbb{R}_+$ and $P = S/|S| \in SO(m)$, we also decompose Ω into its antisymmetric and diagonal part:

$$\Omega = A + \frac{Tr(\Omega)}{m} \otimes \mathbb{I}$$

with $A \in W^{1,p}(\mathbb{B}^n, so(m) \otimes \Lambda^1 \mathbb{R}^n)$.

Let $\Omega^S = S^{-1}dS + S^{-1}\Omega S$; likewise $\Omega^P = P^{-1}dP + P^{-1}\Omega P$ and $A^P = P^{-1}dP + P^{-1}AP$.

We have

$$\begin{aligned}\Omega^S &= \lambda^{-1} P^{-1} (d\lambda \otimes P + \lambda dP + P^{-1} \Omega P) \\ &= d \ln \lambda \otimes \mathbb{I} + \Omega^P.\end{aligned}$$

Decomposing Ω we have

$$\begin{aligned}\Omega^P &= P^{-1} dP + P^{-1} \left(A + \frac{\text{Tr}(\Omega)}{m} \otimes \mathbb{I} \right) P \\ &= A^P + \frac{\text{Tr}(\Omega)}{m} \otimes \mathbb{I},\end{aligned}$$

Thus

$$\Omega^S = A^P + \left(d \ln \lambda + \frac{\text{Tr}(\Omega)}{m} \right) \otimes \mathbb{I}.$$

Clearly, A satisfies all the assumptions on Ω in Theorem 3.1, we can thus find $P \in W^{2,p}(\mathbb{B}^n, SO(m))$ and $\xi \in W^{2,p}(\mathbb{B}^n, so(m) \otimes \Lambda^{n-2}\mathbb{R}^n)$ such that

$$(5.3) \quad \begin{cases} A^P = P^{-1} dP + P^{-1} A P = *d\xi & \text{on } \mathbb{B}^n, \\ d * \xi = 0 & \text{on } \mathbb{B}^n, \\ \xi = 0 & \text{on } \partial \mathbb{B}^n; \end{cases}$$

and such that

$$(5.4a) \quad \|d\xi\|_{W^{1,p}} + \|dP\|_{W^{1,p}} \leq C(n, m) \|A\|_{W^{1,p}} \leq C(n, m) \|\Omega\|_{W^{1,p}}$$

$$(5.4b) \quad \|d\xi\|_{L^p} + \|dP\|_{L^p} \leq C(n, m) \|A\|_{L^p} \leq C(n, m) \|\Omega\|_{L^p},$$

$$(5.4c) \quad \|d\xi\|_{L^n} + \|dP\|_{L^n} \leq C(n, m) \|A\|_{L^n} \leq C(n, m) \|\Omega\|_{L^n}.$$

By Hodge decomposition we can find $\alpha \in W^{2,p}(\mathbb{B}^n)$ and $\beta \in W^{2,p}(\mathbb{B}^n, \Lambda^{n-2}\mathbb{R}^n)$ such that

$$\frac{1}{m} \text{Tr}(\Omega) = d\alpha + *d\beta$$

with $\beta|_{\partial \mathbb{B}^n} = 0$ and $\|d\alpha\|_W^{1,p} + \|d\beta\|_{W^{1,p}} \leq \|\Omega\|_{W^{1,p}}$.

This shows that any λ such that $d \ln \lambda = -d\alpha$ and $\zeta = \xi + \beta \otimes \mathbb{I}$ satisfy (5.1). The estimates (5.2) follow immediately from the estimates on Hodge decomposition and from (5.4). \square

The above theorem is rather simple, but it provides a new interpretation to gradient-like terms $df \otimes \mathbb{I}$ in nonlinear systems – we can incorporate them in antisymmetric expressions and perform Uhlenbeck's decomposition on the resulting TCO_+ matrix of differential forms instead of dealing with both kinds of terms separately. We also can ask a natural question: what is the largest subgroup G of $GL(n)$ that can be used in an analogue of Uhlenbeck's theorem: for any matrix of 1-differential forms $\Omega \in T_{\mathbb{I}}G$ there exists a gauge transformation $P \in G$ such that $\Omega^P = P^{-1} dP + P^{-1} \Omega P$ is coclosed (+integrability estimates on P , Ω^P and their derivatives in terms of Ω)?

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